# Exercise 34

If $a$, $b$, and $c$ are all positive constants and $y(x)$ is a solution of the differential equation $ay'' + by' + cy = 0$, show that $\lim_{x\to\infty} y(x) = 0$.

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**Solution**

This is a linear homogeneous ODE with constant coefficients, so its solutions are of the form $y = e^{rx}$.

$$y = e^{rx} \quad \to \quad y' = re^{rx} \quad \to \quad y'' = r^2 e^{rx}$$

Plug these formulas into the ODE.

$$a(r^2 e^{rx}) + b(re^{rx}) + c(e^{rx}) = 0$$

Divide both sides by $e^{rx}$.

$$ar^2 + br + c = 0$$

Solve for $r$.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \left\{ \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right\}$$

Two solutions to the ODE are

$$\exp\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} x\right) \quad \text{and} \quad \exp\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} x\right).$$

By the principle of superposition, then, the general solution is

$$y(x) = C_1 \exp\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} x\right) + C_2 \exp\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} x\right).$$

If $b^2 - 4ac > 0$, then the square root yields a real number. $-b - \sqrt{b^2 - 4ac} < 0$, and since $\sqrt{b^2 - 4ac} < b$, the coefficients of $x$ in the exponent are negative.

$$\begin{aligned}
\lim_{x\to\infty} y(x) &= C_1 \lim_{x\to\infty} \exp\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a} x\right) + C_2 \lim_{x\to\infty} \exp\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} x\right) \\
&= C_1(0) + C_2(0) \\
&= 0
\end{aligned}$$

If $b^2 - 4ac = 0$, then the square root is zero.

$$\begin{aligned}
\lim_{x\to\infty} y(x) &= C_1 \lim_{x\to\infty} \exp\left(-\frac{b}{2a} x\right) + C_2 \lim_{x\to\infty} \exp\left(-\frac{b}{2a} x\right) \\
&= C_1(0) + C_2(0) \\
&= 0
\end{aligned}$$

If $b^2 - 4ac < 0$, then the square root yields an imaginary number.

$$\begin{aligned}
y(x) &= C_1 \exp\left(\frac{-b - i\sqrt{4ac - b^2}}{2a}x\right) + C_2 \exp\left(\frac{-b + i\sqrt{4ac - b^2}}{2a}x\right) \\
&= C_1 \exp\left(-\frac{b}{2a}x\right) \exp\left(-i\frac{\sqrt{4ac - b^2}}{2a}x\right) + C_2 \exp\left(-\frac{b}{2a}x\right) \exp\left(i\frac{\sqrt{4ac - b^2}}{2a}x\right) \\
&= \exp\left(-\frac{b}{2a}x\right)\left[C_1 \exp\left(-i\frac{\sqrt{4ac - b^2}}{2a}x\right) + C_2 \exp\left(i\frac{\sqrt{4ac - b^2}}{2a}x\right)\right] \\
&= \exp\left(-\frac{b}{2a}x\right)\left[C_1\left(\cos\frac{\sqrt{4ac - b^2}x}{2a} - i\sin\frac{\sqrt{4ac - b^2}x}{2a}\right) + C_2\left(\cos\frac{\sqrt{4ac - b^2}x}{2a} + i\sin\frac{\sqrt{4ac - b^2}x}{2a}\right)\right] \\
&= \exp\left(-\frac{b}{2a}x\right)\left[(C_1 + C_2)\cos\frac{\sqrt{4ac - b^2}x}{2a} + (-iC_1 + iC_2)\sin\frac{\sqrt{4ac - b^2}x}{2a}\right] \\
&= \exp\left(-\frac{b}{2a}x\right)\left(C_3\cos\frac{\sqrt{4ac - b^2}x}{2a} + C_4\sin\frac{\sqrt{4ac - b^2}x}{2a}\right)
\end{aligned}$$

Because of the exponential function, $y(x) \to 0$ in the limit as $x \to \infty$.

$$\begin{aligned}
\lim_{x\to\infty} y(x) &= \lim_{x\to\infty} \exp\left(-\frac{b}{2a}x\right)\left(C_3\cos\frac{\sqrt{4ac - b^2}x}{2a} + C_4\sin\frac{\sqrt{4ac - b^2}x}{2a}\right) \\
&= \left[\lim_{x\to\infty} \exp\left(-\frac{b}{2a}x\right)\right]\left[\lim_{x\to\infty}\left(C_3\cos\frac{\sqrt{4ac - b^2}x}{2a} + C_4\sin\frac{\sqrt{4ac - b^2}x}{2a}\right)\right] \\
&= (0)\left[\lim_{x\to\infty}\left(C_3\cos\frac{\sqrt{4ac - b^2}x}{2a} + C_4\sin\frac{\sqrt{4ac - b^2}x}{2a}\right)\right] \\
&= 0
\end{aligned}$$